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Charged and rotating AdS black holes and their CFT duals

S.W. Hawking* and H.S. Reall†

*University of Cambridge
DAMTP
Silver Street
Cambridge, CB3 9EW
United Kingdom
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Abstract

Black hole solutions that are asymptotic to $AdS_5 \times S^5$ or $AdS_4 \times S^7$ can rotate in two different ways. If the internal sphere rotates then one can obtain a Reissner-Nordstrom-AdS black hole. If the asymptotically AdS space rotates then one can obtain a Kerr-AdS hole. One might expect superradiant scattering to be possible in either of these cases. Superradiant modes reflected off the potential barrier outside the hole would be re-amplified at the horizon, and a classical instability would result. We point out that the existence of a Killing vector field timelike everywhere outside the horizon prevents this from occurring for black holes with negative action. Such black holes are also thermodynamically stable in the grand canonical ensemble. The CFT duals of these black holes correspond to a theory in an Einstein universe with a chemical potential and a theory in a rotating Einstein universe. We study these CFTs in the zero coupling limit. In the first case, Bose-Einstein condensation occurs on the boundary at a critical value of the chemical potential. However the supergravity calculation demonstrates that this is not to be expected at strong coupling. In the second case, we investigate the limit in which the angular velocity of the Einstein universe approaches the speed of light at finite temperature. This is a new limit in which to compare the CFT at strong and weak coupling. We find that the free CFT partition function and supergravity action have the same type of divergence but the usual factor of $4/3$ is modified at finite temperature.

*S.W.Hawking@damtp.cam.ac.uk

†H.S.Reall@damtp.cam.ac.uk

I. INTRODUCTION

Black holes in asymptotically flat space are often thought of as completely dead classically. That is, they can absorb radiation and energy, but not give any out. However, in 1969, Penrose devised a classical process to extract energy from a rotating black hole [1]. This is possible because the horizon is rotating faster than light with respect to the stationary frame at infinity. In other words, the Killing vector k that is time like at infinity is space like on the horizon. The energy-momentum flux vector $J^\mu = T^\mu_\nu k^\nu$ can therefore also be space like, even for matter obeying the dominant energy condition. Thus the energy flow across the future horizon of a rotating black hole can be negative: the Penrose process extracts rotational energy from the hole and slows its spin. This shows that rotating black holes are potentially unstable.

A nice way of extracting rotational energy is to scatter a wave off the black hole [2,3]. Part of the incoming wave will be absorbed, and will change the mass and angular momentum of the hole. By the first law of black hole mechanics

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ, \quad (1.1)$$

the changes in mass and angular momentum determine the change in area of the horizon. The second law

$$dA \geq 0 \quad (1.2)$$

states that the area will increase in classical scattering for fields that obey the dominant energy condition. For a wave of frequency, ω and axial quantum number m the change of mass and the change of angular momentum obey

$$\frac{dM}{dJ} = \frac{\omega}{m}. \quad (1.3)$$

The first and second second laws imply that the change of energy of the black hole is negative when

$$\omega < m\Omega. \quad (1.4)$$

In other words, instead of part of the incident wave being absorbed by the black hole and part reflected back, the reflected wave would actually be stronger than the original incoming wave. Such amplified scattering is called superradiance.

In a purely classical theory, a black hole is won't lose angular momentum to massless fields like gravity. It is a different story, however, with massive fields. A mass term μ for a scalar field, will prevent waves of frequency $\omega < \mu$ from escaping to infinity. Instead they will be reflected by a potential barrier at large radius back into the hole. If they satisfy the condition for superradiance then the waves will be amplified by scattering off the hole. Each time the wave is reflected back, its amplitude will be larger. Thus the wave will grow exponentially and the black hole will lose its angular momentum by a classical process.

One can understand this instability in the following way. In the WKB limit, a mode with $\omega < \mu$ corresponds to a gravitationally bound particle. If its orbital angular velocity

is less than the angular velocity of the black hole then angular momentum and energy will flow from the hole to the particle. Orbits in asymptotically flat space can have arbitrarily long periods so rotating flat space black holes will always lose angular momentum to massive fields, although in practice the rate is very low.

Charged fields scattering off an electrically charged black hole have similar superradiant amplification [4–6]. The condition is now

$$\omega < q\Phi, \quad (1.5)$$

where q is the charge of the field and Φ the electrostatic potential difference between the horizon and infinity. There is, however, an important difference from the rotating case. Black holes with regular horizons, obey a Bogolomony bound, that their charges are not greater than their masses, with equality only in the BPS extreme state. This bound implies that the electro static repulsion between charged black holes, can never be greater than their mutual gravitational attraction. The Bogolomony bound on the charge of a black hole implies that $\Phi \leq 1$ in asymptotically flat space. In a Kaluza-Klein or super symmetric theory, the charges of fields will generally obey the same BPS bound as black holes, with respect to their rest mass i.e. $\mu \geq q$. This means the inequality for superradiance can never be satisfied. One can think of this as a consequence of the fact that the BPS bound implies that gauge repulsions never dominate over gravitational attraction. It means that charged black holes in super symmetric and Kaluza Klein theories are classically stable. The black hole can't lose charge by sending out a charged particle while maintaining the area of the horizon, as it must in a classical process.

So far we have been discussing superradiance and stability of black holes in asymptotically flat space. However, it should also be interesting to study holes in anti-de Sitter space because the AdS/CFT duality [7–9] relates the properties of these holes to thermal properties of a dual conformal field theory living on the boundary of AdS [9,10]. A five dimensional AdS analogue of the Kerr solution was constructed in [11]. Reissner-Nordstrom-AdS (RNAdS) solutions were derived in [12]¹. These holes carry Kaluza-Klein charge coming from the rotation of an internal S^5 . The charged and rotating holes, although appearing rather different in four or five dimensions, therefore appear quite similar from the perspective of ten or eleven dimensional KK theory: one rotates in the AdS space and the other in the internal space. One aim of this paper is to investigate whether these rotating black holes exhibit superradiance and instability and what that implies for the dual CFT. This CFT lives on the conformal boundary of our black hole spacetimes, which is an Einstein universe.

The Kerr-AdS solution is discussed in section II. We find that a superradiant instability is possible only when the Einstein universe on the boundary rotates faster than light. However this can only occur when the black hole is suppressed in the supergravity partition function relative to pure AdS.

¹The phase structure of charged AdS black holes was also studied in [13] but these had non-vanishing scalar fields so they were not RNAdS. The thermodynamics of Kerr-Newman-AdS black holes in *four* dimensions was recently discussed in [14]

We discuss the RNAdS solutions in section III. We point out that it is not possible for the internal S^5 to rotate faster than light in these solutions and therefore superradiance cannot occur, contrary to speculations made in [12]. In particular this means that the extremal black holes, although not supersymmetric, are classically stable. It *is* possible for the internal S^5 to rotate faster than light in $AdS_5 \times S^5$. However such solutions have higher action than the corresponding RNAdS solution (in a grand canonical ensemble) and are therefore suppressed in the supergravity partition function and do not affect the phase structure of the strongly coupled CFT.

A second aim of this paper is to compare the behaviour of the CFT at strong and weak coupling. It was pointed out in [15] that the entropy of the strongly coupled theory (in flat space) is precisely 3/4 that of the free theory - the surprise being that there is no dependence on the t'Hooft parameter λ or the number of colours N . It has also been noticed that the Casimir energy is the same for the free and strongly coupled theories [16]. This suggests that turning up the temperature is similar to turning up the coupling.

We study the boundary CFT using a grand canonical ensemble. In the charged case, this corresponds to turning on a chemical potential for a $U(1)$ subgroup of the $SO(6)$ R-symmetry group. In the rotating case, there are chemical potentials constraining the CFT fields to rotate in the Einstein universe. In the free CFT, a $U(1)$ chemical potential would cause Bose condensation at a critical value. This is not apparent in the strongly coupled theory, which instead exhibits a first order phase transition. Bose condensation has been discussed in the context of spinning branes [17,18] but these discussions have referred to CFTs in flat space, for which the energy of massless fields starts at zero and Bose condensation would occur for any non-zero chemical potential.

The rotating case was studied at high temperature in [11,19]. It was found that the factor of 3/4 relating the free and strongly coupled CFTs persists, even though there are extra dimensionless parameters present that could have affected the result [19]. At high temperature the finite radius of the spatial sections of the Einstein universe is negligible so the theory behaves as if it were in flat space. In the rotating case there is a new limit in which to study the behaviour of the CFT, namely the limit in which the angular velocity of the universe tends to the speed of light. We find that the divergences in the partition functions of the free and strongly coupled CFTs are of the same form at finite temperature in this limit. We also examine how the 3/4 factor is modified at finite temperature.

II. BULK AND BOUNDARY ROTATION

The three parameter Kerr-AdS solution in five dimensions was given in [11]. We shall start by reviewing the properties of this solution, which is expected to be dual to the thermal properties of a strongly coupled CFT in a rotating Einstein universe. We then investigate classical and thermodynamic stability. Finally we calculate the partition function of the free CFT in a rotating Einstein universe in order to compare the properties of the strongly coupled and free theories.

A. Five dimensional Kerr-AdS solution

The five dimensional Kerr-AdS metric is [11]

$$\begin{aligned}
ds^2 = & -\frac{\Delta}{\rho^2} \left(dt - \frac{a_1 \sin^2 \theta}{\Xi_1} d\phi_1 - \frac{a_2 \cos^2 \theta}{\Xi_2} d\phi_2 \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a_1 dt - \frac{(r^2 + a_1^2)}{\Xi_1} d\phi_1 \right)^2 \\
& + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left(a_2 dt - \frac{(r^2 + a_2^2)}{\Xi_2} d\phi_2 \right)^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 \\
& + \frac{(1 + r^2)}{r^2 \rho^2} \left(a_1 a_2 dt - \frac{a_2 (r^2 + a_1^2) \sin^2 \theta}{\Xi_1} d\phi_1 - \frac{a_1 (r^2 + a_2^2) \cos^2 \theta}{\Xi_2} d\phi_2 \right)^2,
\end{aligned} \tag{2.1}$$

where we have scaled the AdS radius to one and

$$\begin{aligned}
\Delta &= \frac{1}{r^2} (r^2 + a_1^2)(r^2 + a_2^2)(1 + r^2) - 2m; \\
\Delta_\theta &= (1 - a_1^2 \cos^2 \theta - a_2^2 \sin^2 \theta); \\
\rho^2 &= (r^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta); \\
\Xi_i &= (1 - a_i^2)
\end{aligned} \tag{2.2}$$

The metric is non-singular outside a horizon at $r = r_+$ provided $a_i^2 < 1$. The angular velocities of the horizon in these coordinates are

$$\Omega'_i = \frac{a_i(1 - a_i^2)}{r_+^2 + a_i^2} \tag{2.3}$$

The corotating Killing vector field is

$$\chi = \frac{\partial}{\partial t} + \Omega'_1 \frac{\partial}{\partial \phi_1} + \Omega'_2 \frac{\partial}{\partial \phi_2}, \tag{2.4}$$

which is tangent to the null geodesic generators of the horizon. These coordinates are not well-suited to demonstrating the asymptotically AdS nature of this solution. A more appropriate set of coordinates is defined as follows [11]

$$\begin{aligned}
T &= t; \\
\Xi_1 y^2 \sin^2 \Theta &= (r^2 + a_1^2) \sin^2 \theta; \\
\Xi_2 y^2 \cos^2 \Theta &= (r^2 + a_2^2) \cos^2 \theta; \\
\Phi_i &= \phi_i + a_i t.
\end{aligned} \tag{2.5}$$

In these coordinates, the angular velocities become

$$\Omega_i = \frac{a_i(1 + r_+^2)}{r_+^2 + a_i^2} \tag{2.6}$$

and the corotating Killing vector field is

$$\chi = \frac{\partial}{\partial T} + \Omega_1 \frac{\partial}{\partial \Phi_1} + \Omega_2 \frac{\partial}{\partial \Phi_2}. \tag{2.7}$$

The conformal boundary of the spacetime is an Einstein universe $R \times S^3$ with metric

$$ds^2 = -dT^2 + d\Theta^2 + \sin^2 \Theta d\Phi_1^2 + \cos^2 \Theta d\Phi_2^2. \quad (2.8)$$

The action of the hole relative to an AdS background is calculated by considering the Euclidean section of the hole obtained by analytically continuing the time coordinate. To avoid a conical singularity it is necessary to identify $(t, y, \Theta, \Phi_1, \Phi_2)$ with $(t + i\beta, y, \Theta, \Phi_1 + i\beta\Omega_1, \Phi_2 + i\beta\Omega_2)$ where

$$\beta = \frac{4\pi(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{r_+^2 \Delta'(r_+)}, \quad (2.9)$$

The same identifications must be made in the AdS background in order to perform the matching. The action relative to AdS is [11]

$$I = -\frac{\pi\beta(r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 - 1)}{8G_5 r_+^2 (1 - a_1^2)(1 - a_2^2)}, \quad (2.10)$$

where G_5 is Newton's constant in five dimensions. The action is negative only for $r_+ > 1$. The boundary Einstein universe inherits the above identifications from the bulk. The usual arguments [20] then show that this identified Einstein universe is the appropriate background for path integrals defining thermal partition functions at temperature

$$T = \frac{1}{\beta} = \frac{2r_+^6 + (1 + a_1^2 + a_2^2)r_+^4 - a_1^2 a_2^2}{2\pi r_+ (r_+^2 + a_1^2)(r_+^2 + a_2^2)} \quad (2.11)$$

and with chemical potentials Ω_i for the angular momenta J_i of matter fields in the Einstein universe. Matter is therefore constrained to rotate in the Einstein universe; this is what is meant by saying that the universe is rotating. The mass and angular momenta of the black hole (using the coordinate (T, Φ_i)) are [11]

$$M = \frac{3\pi m}{4(1 - a_1^2)(1 - a_2^2)}, \quad J_i = \frac{\pi a_i m}{2(1 - a_i^2)(1 + r_+^2)}. \quad (2.12)$$

B. Stability of Kerr-AdS

In an asymptotically flat Kerr background there is a unique (up to normalization) Killing vector field timelike near infinity i.e. $k = \partial/\partial t$. Near the horizon there is an ergosphere - a region where k becomes spacelike, and energy extraction through superradiance becomes possible for modes satisfying equation 1.4. In AdS, superradiance would correspond to an instability of the hole. This is because superradiant modes would be reflected back towards the hole by a potential barrier (in the case of massive fields) or boundary conditions at infinity (for massless fields) and reamplified at the horizon before being scattered again. The hole would be classically unstable and would lose angular momentum to a cloud of particles orbiting it. The spectrum of fields in AdS is discrete, and one might expect the threshold value of Ω for superradiance to be given by the minimum of $\omega/|m|$ for fields in the black hole background. However the presence of a black hole changes the spectrum from

discrete to continuous (since regularity at the origin is no longer required) and it is not clear whether a positive lower bound exists. Fortunately there is a simple argument that demonstrates the stability of Kerr-AdS for $|\Omega_i| < 1$.

In Kerr-AdS, if $|\Omega_i| < 1$ then the corotating Killing vector field χ is timelike everywhere outside the horizon, so there is a corotating frame that exists all the way out to infinity (in contrast with the situation in flat space, where any rigidly rotating frame, will necessarily move faster than light far from the axis of rotation). The energy-momentum vector in this frame is $J^\mu = T^\mu_\nu \chi^\nu$. If the matter obeys the dominant energy condition [21] then this is non-spacelike everywhere outside the horizon. Let Σ be a spacelike hypersurface from the horizon to infinity with normal n_μ . The total energy of matter on Σ is

$$E = - \int_{\Sigma} d^4x \sqrt{h} n_\mu J^\mu, \quad (2.13)$$

where h is the determinant of the induced metric on Σ . The integrand is everywhere non-positive so $E \geq 0$. The normal to the horizon is χ_μ , so the energy flux density across the horizon is $J^\mu \chi_\mu$, which is non-positive. If suitable boundary conditions are imposed then energy will not enter the spacetime from infinity. This means that if E is evaluated on another surface Σ' lying to the future of Σ then

$$E(\Sigma') \leq E(\Sigma), \quad (2.14)$$

that is, E is non-increasing function that is bounded below by zero. Energy cannot be extracted from the hole: it is classically stable.

When $\Omega_i^2 > 1$, the corotating Killing vector field *does* become spacelike in a region near infinity: this region rotates faster than light. The above argument then breaks down and an instability may occur. There are two different limits in which $\Omega_i^2 \rightarrow 1$ [11]. The first is $a_i^2 \rightarrow 1$, which makes the metric become singular. The second is $r_+^2 \rightarrow a_i$ for which the metric remains regular. In fact there is a range of $r_+^2 < a_i$ for which $\Omega_i^2 > 1$. However since $a_i < 1$, these black holes all have $r_+ < 1$ and hence have positive action. They are therefore suppressed relative to AdS in the supergravity partition function, so even if these holes are unstable, the instability will not affect the phase structure of the CFT, although it may be of interest in its own right.

We have demonstrated the absence of a classical instability when $|\Omega_i| < 1$. However we have not yet discussed thermodynamic stability. In order to uniquely define the grand canonical ensemble, the Legendre transformation from the extensive variables (M, J_1, J_2) to the intensive variables (T, Ω_1, Ω_2) must be single-valued. If this Legendre transformation becomes singular then the grand-canonical ensemble becomes ill-defined. It is straightforward to calculate the determinant of the jacobian:

$$\det \frac{\partial(T, \Omega_1, \Omega_2)}{\partial(E, J_1, J_2)} = \det \frac{\partial(T, \Omega_1, \Omega_2)}{\partial(r_+, a_1, a_2)} / \det \frac{\partial(E, J_1, J_2)}{\partial(r_+, a_1, a_2)}. \quad (2.15)$$

The denominator vanishes if, and only if,

$$2(1 - a_1^2 a_2^2) r_+^6 + (1 + a_1^2(2 - a_1^2) + a_2^2(2 - a_2^2) + a_1^2 a_2^2(3 - a_1^2 a_2^2)) r_+^4 + 2a_1^2 a_2^2(2 + a_1^2 a_2^2) r_+^2 - a_1^2 a_2^2(1 - a_1^2 - a_2^2 - 3a_1^2 a_2^2) = 0. \quad (2.16)$$

The right hand side can be written as

$$(1 - a_1^2 a_2^2) \left[2r_+^6 + (1 + a_1^2 + a_2^2)r_+^4 - a_1^2 a_2^2 \right] + \dots \quad (2.17)$$

where the ellipsis denotes a group of terms that is easily seen to be positive. The quantity in square brackets must also be positive in order for a black hole solution to exist (as can be seen from equation 2.11). Therefore equation 2.16 cannot be satisfied. The numerator in equation 2.15 vanishes if, and only if,

$$2r_+^6 - (1 + a_1^2 + a_2^2)r_+^4 + a_1^2 a_2^2 = 0. \quad (2.18)$$

The right hand side is positive for $r_+ > 1$. It has a negative minimum at a value of r_+ between 0 and 1 and is positive at $r_+ = 0$ so there must be two roots between 0 and 1. Let r_0 denote the larger of these two roots. Black holes with $r_+ > r_0$ are locally thermodynamically stable. However only those with $r_+ > 1$ have negative action, so the holes with $r_0 < r_+ < 1$ are only metastable. The requirement of an invertible Legendre transformation therefore does not affect the phase structure obtained from the action calculation. Four dimensional Kerr-AdS black holes behave in the same way [14].

C. Free CFT in a rotating Einstein universe

The high temperature limit of free fields in a rotating Einstein universe was recently investigated in [11,19]. The usual factor of 4/3 between the strongly coupled and free CFTs was found to persist [19]. We wish to investigate a different limit, namely $\Omega_i \rightarrow \pm 1$ at *finite* temperature. At finite temperature, the finite size of the S^3 spatial sections of the Einstein universe becomes significant. To compute the partition function we need to know the spectrum of the CFT fields in the Einstein universe.

The Einstein universe has isometry group $R \times SO(4) = R \times SU(2) \times SU(2)$, so we may classify representations of the isometry group according to the Casimirs (ω, j_L, j_R) of R and the two $SU(2)$'s. The little group is $SO(3) = SU(2)/Z_2$. The generators of this group are $J_i = J_i^{(L)} + J_i^{(R)}$, where $J_i^{(L)}$ and $J_i^{(R)}$ are the generators of the two $SU(2)$ groups. Therefore the $SO(3)$ content of the representations of the isometry group is given by angular momentum addition. The irreducible representation (ω, j_L, j_R) will give a sum of irreducible representations of the little group, with $|j_L - j_R| \leq j \leq j_L + j_R$. The lowest eigenvalue $j = |j_L - j_R|$ is regarded as the spin. Therefore irreducible representations of the form $(\omega, j, j \pm s)$ describe particles of spin s . Parity invariance is obtained by taking the direct sum $(\omega, j, j + s) + (\omega, j + s, j)$. These representations may be promoted to representations of the conformal group provided ω is suitably related to j and s . The allowed values of ω can be obtained by solving conformally invariant wave equations on the Einstein universe. Alternatively they can be solved on AdS_4 , which is conformal to half of the Einstein universe. This was done in [22]. The scalar modes on AdS_4 can be extended to modes on the Einstein universe. There are two different complete sets of modes on AdS_4 however both sets are required for completeness on the Einstein universe. The same happens for modes of higher spin.

The scalar modes form the representations (j, j) of $SU(2) \times SU(2)$. The energy eigenvalues are given by $\omega = J + 1$ where $J = 2j$. The spin-1/2 modes form the representations

$(j, j + 1/2) + (j + 1/2, j)$ and have $\omega = J + 1$ where $J = 2j + 1/2$. The spin-1 modes form the representations $(j, j + 1) + (j + 1, j)$ with $\omega = J + 1$ and $J = 2j + 1$. In all cases the allowed values of j are $0, 1/2, 1, \dots$. We have not taken account of the Casimir energy of the fields because we have measured all energies relative to AdS rather than using the boundary counterterm method [16] to calculate the supergravity action.

The Killing vector fields of the Einstein universe form a representation of the Lie algebra of the isometry group with $\partial/\partial\Phi_1 = J_3^{(L)} - J_3^{(R)}$ and $\partial/\partial\Phi_2 = J_3^{(L)} + J_3^{(R)}$. Thus the quantum numbers corresponding to rotations in the Φ_1 and Φ_2 directions are $m_L - m_R$ and $m_L + m_R$ respectively.

We can now compute the partition functions for the CFT fields. In the grand canonical ensemble, these are given by

$$\log Z = \mp \sum \log(1 \mp e^{-\beta(\omega - \Omega_1(m_L - m_R) - \Omega_2(m_L + m_R))}), \quad (2.19)$$

where the upper sign is for the bosons and the lower sign for the fermions. Using the energy levels given above, the partition function for a conformally coupled scalar field is given by

$$\begin{aligned} \log Z_0 &= - \sum_{J=0}^{\infty} \sum_{m_L=-J/2}^{J/2} \sum_{m_R=-J/2}^{J/2} \log \left(1 - e^{-\beta(J+1-\Omega_1(m_L-m_R)-\Omega_2(m_L+m_R))} \right) \\ &= - \sum_{J=0}^{\infty} \sum_{m_L=-J/2}^{J/2} \sum_{m_R=-J/2}^{J/2} \log \left(1 - e^{-\beta(J+1-\Omega_+m_L-\Omega_-m_R)} \right), \end{aligned} \quad (2.20)$$

where $\Omega_{\pm} = \Omega_1 \pm \Omega_2$, the J -summation runs over integer values and we have reversed the order of the m_R summation. The partition function for a conformally coupled spin-1/2 field is given by

$$\log Z_{1/2} = \sum_{J=1/2}^{\infty} \sum_{m_L=-(J+1/2)/2}^{(J+1/2)/2} \sum_{m_R=-(J-1/2)/2}^{(J-1/2)/2} \log \left(1 + e^{-\beta(J+1-\Omega_+m_L-\Omega_-m_R)} \right) + (\Omega_+ \leftrightarrow \Omega_-), \quad (2.21)$$

where the J -summation runs over half odd integer values. The first term comes from the $(j + 1/2, j)$ representations and the second from the $(j, j + 1/2)$ ones. The partition function for a conformally coupled spin-1 field is given by

$$\log Z_1 = - \sum_{J=1}^{\infty} \sum_{m_L=-(J+1)/2}^{(J+1)/2} \sum_{m_R=-(J-1)/2}^{(J-1)/2} \log \left(1 - e^{-\beta(J+1-\Omega_+m_L-\Omega_-m_R)} \right) + (\Omega_+ \leftrightarrow \Omega_-), \quad (2.22)$$

where the J -summation runs over integer values.

When β is small, the sums in the above expressions may be replaced by integrals. Doing so, one recovers the results of [19]. For general β we instead expand the logarithms as power series. This gives

$$\log Z_0 = \sum_{J=0}^{\infty} \sum_{m_L=-J/2}^{J/2} \sum_{m_R=-J/2}^{J/2} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\beta(J+1-\Omega_+m_L-\Omega_-m_R)}. \quad (2.23)$$

We now interchange the orders of the n and J summations² The summations over m_L, m_R and J can then be done (they are all geometric series). One obtains

$$\log Z_0 = \sum_{n=1}^{\infty} \frac{e^{\beta n} (e^{2\beta n} - 1)}{n (e^{\beta n(1-\Omega_1)} - 1) (e^{\beta n(1+\Omega_1)} - 1) (e^{\beta n(1-\Omega_2)} - 1) (e^{\beta n(1+\Omega_2)} - 1)}. \quad (2.24)$$

Similar calculations give

$$\log Z_{1/2} = \sum_{n=1}^{\infty} \frac{(-)^{n+1} 4e^{3\beta n/2} (e^{\beta n} - 1) \cosh(\beta n \Omega_1/2) \cosh(\beta n \Omega_2/2)}{n (e^{\beta n(1-\Omega_1)} - 1) (e^{\beta n(1+\Omega_1)} - 1) (e^{\beta n(1-\Omega_2)} - 1) (e^{\beta n(1+\Omega_2)} - 1)} \quad (2.25)$$

and

$$\log Z_1 = \sum_{n=1}^{\infty} \frac{4 (e^{\beta n} \cosh(\beta n \Omega_1) - 1) (e^{\beta n} \cosh(\beta n \Omega_2) - 1) + 2 (e^{2\beta n} - 1)}{n (e^{\beta n(1-\Omega_1)} - 1) (e^{\beta n(1+\Omega_1)} - 1) (e^{\beta n(1-\Omega_2)} - 1) (e^{\beta n(1+\Omega_2)} - 1)}. \quad (2.26)$$

Note that all of these diverge as $\Omega_i \rightarrow \pm 1$. At first sight this looks like Bose-Einstein condensation (since Ω_i is a chemical potential) but this is misleading. The divergence does not arise from the lowest bosonic energy level but from summing over all of the modes (in particular the modes with largest $\Omega_+ m_L + \Omega_- m_R$ for each J [11]). Furthermore the fermion partition function also diverges, so this is certainly not a purely bosonic effect.

The particle content of the $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory is N^2 gauge bosons, $4N^2$ Majorana fermions and $6N^2$ scalars. Adding the appropriate contributions from these fields, one obtains the following asymptotic behaviour for the free CFT as $\Omega_1 \rightarrow \pm 1$:

$$\log Z \approx \frac{2N^2}{\beta(1-\Omega_1^2)} \sum_{n=1}^{\infty} \frac{(\cosh(\beta n \Omega_2/2) + (-)^{n+1})^2}{n^2 \sinh(\beta n(1-\Omega_2)/2) \sinh(\beta n(1+\Omega_2)/2)}, \quad (2.27)$$

and if we now let $\Omega_2 \rightarrow \pm 1$ then

$$\log Z \approx \frac{8N^2}{\beta^2(1-\Omega_1^2)(1-\Omega_2^2)} \sum_{n=1}^{\infty} \frac{(\cosh(\beta n/2) + (-)^{n+1})^2}{n^3 \sinh(\beta n)}. \quad (2.28)$$

The divergences as $\Omega_i \rightarrow 1$ are of the same form at all temperatures. We are interested in comparing these divergences as for the free and strongly coupled CFTs. The partition function for the strongly coupled CFT is given by the bulk supergravity partition function. For $r_+ > 1$ this is dominated by the Kerr-AdS solution. To compare with the free CFT results we introduce the stringy parameters. The five dimensional Newton constant is related to the ten dimensional one by $1/G_5 = \pi^3/G_{10}$, where the numerator is simply the volume of the internal S^5 . We are still using units for which the AdS length scale is unity, which means that $\lambda^{1/4} l_s = 1$ when we appeal to the AdS/CFT correspondence. The ten-dimensional Newton constant is related to the CFT parameters by $G_{10} = \pi^4/(2N^2)$ so $G_5 = \pi/(2N^2)$. The supergravity action can then be written

²This can be justified by cutting off the J summation at $J = J_0$, proceeding as described in the text and letting $J_0 \rightarrow \infty$ at the end.

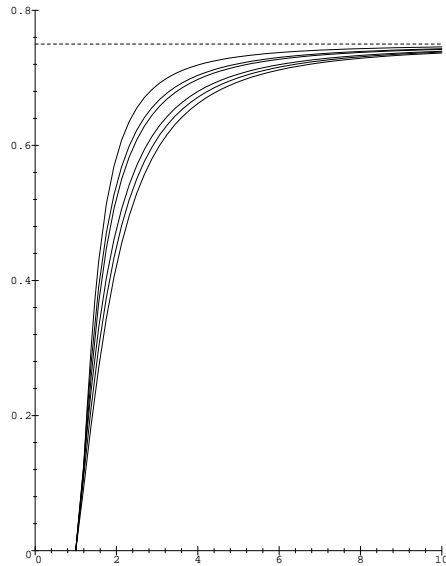


FIG. 1. Ratio of $\log Z$ for strongly coupled CFT to $\log Z$ for free CFT as a function of r_+ . From bottom to top the curves are: $a_1 = a_2 = 0$; $a_1 = 0, a_2 = 0.5$; $a_1 = 0.5, a_2 = 0.5$; $a_1 \rightarrow 1, a_2 = 0$; $a_1 \rightarrow 1, a_2 = 0.5$; $a_1, a_2 \rightarrow 1$.

$$I = -\frac{N^2 \beta (r_+^2 + a_1^2)(r_+^2 + a_2^2)(r_+^2 - 1)}{4r_+^2(1 - a_1^2)(1 - a_2^2)}. \quad (2.29)$$

Recall that in the bulk theory there are two ways to take $\Omega_i \rightarrow 1$. However one of these corresponds to a black hole suppressed relative to AdS. We must therefore use the other limit, namely $a_i \rightarrow 1$. It is convenient to use r_+ and a_i instead of β and Ω_i when comparing the partition functions for the strongly coupled and free CFTs. The divergent factors in the free CFT are

$$\frac{1}{1 - \Omega_i^2} = \frac{(r_+^2 + a_i^2)^2}{(r_+^4 - a_i^2)(1 - a_i^2)}, \quad (2.30)$$

so both the strongly coupled and free CFTs have divergences proportional to $(1 - a_i^2)^{-1}$ in $\log Z$ as $a_i \rightarrow 1$. This generalizes the high temperature results of [11,19].

The ratio

$$f(r_+, a_1, a_2) \equiv \frac{\log Z(\text{strong})}{\log Z(\text{free})} = -\frac{I}{\log Z(\text{free})}, \quad (2.31)$$

is plotted as a function of r_+ for several cases of interest in figure 1. At large r_+ , $\beta \approx 0$, so the radius of the S^3 is much larger than that of the S^1 of the Euclidean time direction. The theory behaves as if it were in flat space. This is why one recovers the flat space result [15] $f(\infty, 0, 0) = 3/4$. The surprise pointed out in [19] is that this is independent of a_i i.e. $f(\infty, a_1, a_2) = 3/4$. We have been studying a different limit, namely $a_i \rightarrow 1$. A

priori there is no reason why this should commute with the high temperature limit but it is straightforward to use the above expressions to show that this is in fact the case, so all of the curves in figure 1 approach $3/4$ at large r_+ . At lower temperatures, there is still not much dependence of f on a_i . What is perhaps more surprising is how rapidly f approaches $3/4$: $f > 0.7$ for $r_+ = 5.5$, which corresponds to $\beta \approx 0.58$ (for all a_i) so the radii of the time and spatial directions are of the same order of magnitude and one might have expected finite size effects to be more important than they appear.

D. The four dimensional case

The AdS/CFT correspondence relates the worldvolume theory of N M2-branes in the large N limit to eleven dimensional supergravity on S^7 . Four dimensional Kerr-AdS black holes are expected to be dual to the worldvolume CFT in a rotating three dimensional Einstein universe. For completeness we present the free CFT results for this case. The CFT is a free supersingleton field theory [23]. There are eight real scalar fields and eight Majorana spin- $1/2$ fields. The energy levels of these fields are $\omega = j + 1/2$ where $j = 0, 1, \dots$ for the scalars and $j = 1/2, 3/2, \dots$ for the fermions [24]. The partition functions can be evaluated as above. For the scalars one obtains

$$\begin{aligned} \log Z_0 &= - \sum_{j=0}^{\infty} \sum_{m=-j}^j \log(1 - e^{-\beta(j+1/2-m\Omega)}) \\ &= \sum_{n=1}^{\infty} \frac{\cosh(\beta n/2)}{2n \sinh(\beta n(1-\Omega)/2) \sinh(\beta n(1+\Omega)/2)}, \end{aligned} \quad (2.32)$$

and for the fermions,

$$\begin{aligned} \log Z_{1/2} &= \sum_{j=1/2}^{\infty} \sum_{m=-j}^j \log(1 + e^{-\beta(j+1/2-m\Omega)}) \\ &= \sum_{n=1}^{\infty} \frac{(-)^{n+1} \cosh(\beta n/2)}{2n \sinh(\beta n(1-\Omega)/2) \sinh(\beta n(1+\Omega)/2)}. \end{aligned} \quad (2.33)$$

Thus the partition function for the free CFT of an M2-brane is

$$\log Z = 8 \sum_{n \text{ odd}} \frac{\cosh(\beta n/2)}{n \sinh(\beta n(1-\Omega)/2) \sinh(\beta n(1+\Omega)/2)}. \quad (2.34)$$

At high temperature, one obtains

$$\log Z_0 \approx \frac{2\zeta(3)}{\beta^2(1-\Omega^2)}, \quad \log Z_{1/2} \approx \frac{3\zeta(3)}{2\beta^2(1-\Omega^2)}. \quad (2.35)$$

If $|\Omega| \rightarrow 1$ at finite temperature then

$$\log Z_0 \approx \frac{1}{\beta(1-\Omega^2)} \sum_{n=1}^{\infty} \frac{1}{n^2 \sinh(\beta n/2)}, \quad (2.36)$$

$$\log Z_{1/2} \approx \frac{1}{\beta(1-\Omega^2)} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^2 \sinh(\beta n/2)}, \quad (2.37)$$

and

$$\log Z \approx \frac{16}{\beta(1-\Omega^2)} \sum_{n \text{ odd}} \frac{1}{n^2 \sinh(\beta n/2)}. \quad (2.38)$$

The divergence is of the same form as that obtained from the bulk supergravity action in the limit $|a| \rightarrow 1$ [11].

III. BULK CHARGE AND BOUNDARY CHEMICAL POTENTIAL

It was shown in [12] how to obtain Einstein-Maxwell theory with a negative cosmological constant from KK reduction of IIB supergravity on S^5 . The reduction ansatz for the metric is³

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \sum_{i=1}^3 [d\mu_i^2 + \mu_i^2 (d\phi_i + A_\mu dx^\mu)^2], \quad (3.1)$$

where $g_{\mu\nu}$ is a five dimensional metric, μ_i are direction cosines on the S^5 (so $\sum_{i=1}^3 \mu_i^2 = 1$) and the ϕ_i are rotation angles on S^5 in three orthogonal planes (when embedded in R^6). Non-vanishing A_μ corresponds to rotating the S^5 by equal amounts in each of these three planes, and gives a Maxwell electromagnetic potential in five dimensions after KK reduction. The ansatz for the Ramond-Ramond 5-form is given in [12].

A. $AdS_5 \times S^5$ with electrostatic potential

The simplest solution of the Einstein-Maxwell system with negative cosmological constant is AdS_5 with metric

$$ds^2 = -U(r)dt^2 + U(r)^{-1}dr^2 + r^2 d\Omega_3^2 \quad (3.2)$$

where

$$U(r) = 1 + r^2 \quad (3.3)$$

and a constant electrostatic potential $A = -\Phi dt$ with $\Phi = \text{const}$. Increasing Φ corresponds to increasing the angular velocity of the internal S^5 . A point at fixed μ_i and ϕ_i on the S^5 moves on an orbit of $k = \partial/\partial t$. This has norm

$$k^2 = -U(r) + \Phi^2, \quad (3.4)$$

³We have rescaled the electromagnetic potential relative to that of [12].

so k will be spacelike in a region near $r = 0$ when $\Phi^2 > 1$. This means that the S^5 rotates faster than light near the origin in AdS_5 when Φ is large. The t -direction becomes spacelike and an internal direction becomes timelike, indicating an instability. If a BPS particle were added to this solution in the grand ensemble then, near the origin, its negative electric potential energy would exceed its rest mass, so the most probable configuration would involve an infinite number of particles.

In the AdS/CFT correspondence, a bulk electromagnetic potential A couples to a conserved current of the boundary theory [8,9]. In our case, the electromagnetic potential is associated with the $U(1)$ obtained by taking equal charges for the three $U(1)$ groups in the $U(1)^3$ Cartan subalgebra of the $SO(6)$ KK gauge group. The CFT current is therefore obtained by taking the same $U(1)$ subgroup of the $U(1)^3$ Cartan subalgebra of the $SU(4)$ R-symmetry group of the boundary CFT. The coupling of the bulk gauge field to the boundary current is $-A_i j^i$, where

$$\begin{aligned} j_i &= r^2 \sum_{k=1}^3 \mu_k^2 \partial_i \phi_k + \text{fermions} \\ &= \sum_{k=1}^3 \left(X^{2k-1} \partial_i X^{2k} - X^{2k} \partial_i X^{2k-1} \right) + \text{fermions} \end{aligned} \quad (3.5)$$

where X^k are the usual scalar fields of the $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory and there is a suppressed sum over N . The fermionic contribution should be straightforward to calculate although we shall not do so.

Taking $A = -\Phi dt$ corresponds to turning on a chemical potential Φ for the $U(1)$ charge in the boundary theory. In the free CFT, Bose-Einstein condensation will result when this chemical potential equals the lowest bosonic energy level, which is $\omega = 1$ (see section II C). Thus BE condensation occurs in the free CFT when $\Phi = \pm 1$ (the two signs refer to particles and anti-particles respectively). This is precisely the critical value of Φ for which the internal sphere rotates at the speed of light.

B. Reissner-Nordstrom-AdS black holes

Solutions of type IIB supergravity describing Reissner-Nordstrom-AdS black holes with an internal S^5 were given in [12]. The five dimensional metric can be written

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_3^2, \quad (3.6)$$

with

$$\begin{aligned} V(r) &= 1 - \frac{M}{r^2} + \frac{Q^2}{r^4} + r^2 \\ &= \left(1 - \frac{r_+^2}{r^2}\right) \left(1 - \frac{r_-^2}{r^2}\right) (1 + r^2 + r_+^2 + r_-^2), \end{aligned} \quad (3.7)$$

where M and Q measure the black hole's mass and charge and r_{\pm} are the outer and inner horizon radii. The electromagnetic potential in a gauge regular on the outer horizon is

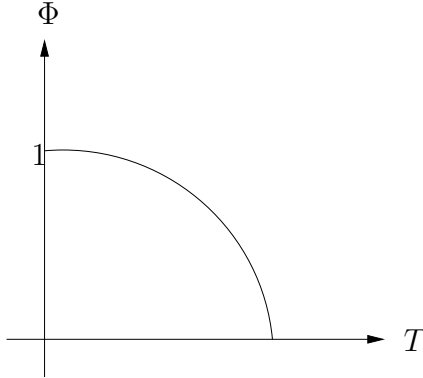


FIG. 2. Phase diagram for Reissner-Nordstrom-AdS. AdS is preferred in the region near the origin.

$$A = (\Phi(r_+) - \Phi(r))dt. \quad (3.8)$$

where

$$\Phi(r) = \frac{Q}{r^2}. \quad (3.9)$$

Once again we can compute the norm of the Killing vector field $k = \partial/\partial t$ with respect to the 10-dimensional metric. This is

$$\begin{aligned} k^2 &= -V(r) + \frac{Q^2}{r_+^4} \left(1 - \frac{r_+^2}{r^2}\right)^2 \\ &= -\left(1 - \frac{r_+^2}{r^2}\right) \left[r^2 \left(1 - \frac{r_-^2}{r^2}\right) + \left(1 - \frac{r_-^2}{r_+^2}\right) (1 + r_+^2 + r_-^2) \right] \end{aligned} \quad (3.10)$$

and this is negative for $r > r_+$. Hence the internal S^5 never rotates faster than light outside the black hole: there is an everywhere timelike Killing vector field outside the hole. The stability argument we used for Kerr-AdS can be therefore also be applied in this case to conclude that energy extraction from RNAdS black holes is impossible.

The action I of the black hole relative to AdS is [12]

$$I = \frac{\pi}{8G_5} \beta (r_+^2 (1 - \Phi(r_+)^2) - r_+^4) \quad (3.11)$$

where the inverse temperature is

$$\beta = \frac{2\pi r_+}{1 - \Phi(r_+)^2 + 2r_+^2}. \quad (3.12)$$

This action can be related to the thermal partition function of the strongly coupled gauge theory on the boundary [9,10]. We are interested in the partition function in the grand canonical ensemble, for which the chemical potential and temperature are fixed on the boundary. The phase diagram was given in [12] and reproduced in figure 2. There is a region near the origin of the $\Phi - T$ plane for which I is positive so AdS is preferred over the black hole. Everywhere else, I is negative so the hole is preferred. A first order phase transition occurs when I changes sign. Note that the internal sphere does not reach the speed

of light anywhere in this diagram. The closest one can get is to let T tend to zero whilst increasing Φ to the critical value in AdS. As soon as the critical value is reached, an extreme black hole of vanishing horizon radius becomes preferred over pure AdS. Thermodynamic stability of RNAdS was discussed in [25]. In the grand canonical ensemble, it was found that black holes with positive action are stable.

This phase diagram is very different from that of the free boundary CFT, which only has a phase transition at the critical value of Φ . The strongly coupled CFT does not exhibit a phase transition as the chemical potential is increased at high temperature, unlike the free CFT. Thus at finite chemical potential, the thermal partition functions of the free and strongly coupled CFTs in an Einstein universe differ by much more than a simple numerical factor, even at high temperature.

In four dimensions the situation is identical. The lowest bosonic energy level is $\omega = 1/2$ (see section IID) so Bose condensation in the free field theory occurs at $\Phi = 1/2$, which is the critical value for the internal sphere in $AdS_4 \times S^7$ to rotate at the speed of light (the KK ansatz for the four dimensional case was given in [12]). The phase structure of the strongly coupled theory is qualitatively identical to the one in figure 2 except that the phase transition occurs at $\Phi = 1/2$ on the $T = 0$ axis.

IV. DISCUSSION

We have studied the stability of rotating asymptotically $AdS_5 \times S^5$ and $AdS_4 \times S^7$ solutions of supergravity. A classical instability can occur if either the boundary of the AdS space or the internal S^5 rotates faster than light. However this occurs only when the solution has positive action relative to AdS and is therefore suppressed in the supergravity partition function. Reissner-Nordstrom-AdS solutions do not exhibit a superradiant instability but small Kerr-AdS solutions may do, although a proof would involve studying wave equations in Kerr-AdS. We have also studied quantum local thermodynamic stability and found that the solutions that are not locally stable have positive action. AdS space is preferred in the domain of the black hole parameters for which the holes are locally unstable. This is to be contrasted with the charged black holes of [13], for which there was a region of parameter space in the grand canonical ensemble where the black holes were preferred over AdS but not locally stable.

We have compared the strongly coupled coupled and free boundary CFTs in an Einstein universe. When the Einstein universe rotates, we find that the free and strongly coupled theories have the same type of divergence as the angular velocities approach the speed of light at finite temperature. The factor of $3/4$ relating the partition functions is recovered at high temperature in the Einstein universe since then the radius of curvature of the S^3 spatial sections is negligible compared with the radius of curvature of the Euclidean time direction. That this factor is independent of the angular velocities at high temperature was noticed in [19]; we have found that it does not vary greatly with angular velocity at lower temperatures either.

Free field theory in the Einstein universe is not a good guide to the properties of the strongly coupled theory at finite $U(1)$ chemical potential since the former would undergo Bose condensation at a critical chemical potential whereas the latter does not. Studying this in the Einstein universe allows us to avoid the problems associated with chemical potentials

in CFTs in flat space. The critical chemical potential at which Bose condensation occurs is the value of the potential for which the internal sphere in $AdS_5 \times S^5$ rotates at the speed of light. However the phase transition in the strongly coupled theory only occurs at this value in the limit of zero temperature.

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